Eigenvalues and Homology of Flag Complexes and Vector Representations of Graphs

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Abstract

The flag complex of a graph $G = (V, E)$ is the simplicial complex $X(G)$ on the vertex set V whose simplices are subsets of V which span complete subgraphs of G. We study relations between the first eigenvalues of successive higher Laplacians of $X(G)$. One consequence is the following

Theorem: Let $\lambda_2(G)$ denote the second smallest eigenvalue of the Laplacian of G. If $\lambda_2(G) > \frac{k}{k+1}|V|$ then $\tilde{H}^k(X(G); \mathbb{R}) = 0$.

Applications include a lower bound on the homological connectivity of the independent sets complex $\mathrm{I}(G)$, in terms of a new graph domination parameter $\Gamma(G)$ defined via certain vector representations of G. This in turns implies a Hall type theorem for systems of disjoint representatives in hypergraphs.

Math Subject Classification: 13F55, 05C69.
Keywords: Flag complexes, homology, domination in graphs.

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1 Introduction

Let $G=(V,E)$ be a graph with $|V|=n$ vertices. The *Laplacian* of G is the $V\times V$ positive semidefinite matrix L_G given by

$$L_G(u,v)=\begin{cases}\deg(u) & u=v\\ -1 & uv\in E\\ 0 & \text{otherwise}\end{cases}$$

Let $0=\lambda_1(G)\leq\lambda_2(G)\leq\cdots\leq\lambda_n(G)$ denote the eigenvalues of L_G. The second smallest eigenvalue $\lambda_2(G)$, called the *spectral gap*, is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of G and the convergence rate of a random walk on G (see e.g. [6]). The *Flag Complex* of G is the simplicial complex $X(G)$ on the vertex set V whose simplices are all subsets $\sigma\subset V$ which form a complete subgraph of G. Topological properties of $X(G)$ play key roles in recent results in matching theory (see below).
In this paper we study relations between $\lambda_2(G)$, the cohomology of $X(G)$, and a new graph domination parameter $\Gamma(G)$ which is defined via certain vector representations of G. As an application we obtain a Hall type theorem for systems of disjoint representatives in families of hypergraphs.

For $k\geq -1$ let $C^k(X(G))$ denote the space of real valued simplicial k-cochains of $X(G)$ and let $d_k: C^k(X(G))\rightarrow C^{k+1}(X(G))$ denote the coboundary operator. For $k\geq 0$ define the reduced k-dimensional Laplacian of $X(G)$ by $\Delta_k=d_{k-1}d_{k-1}^*+d_k^*d_k$ (see section 2 for details). Let $\mu_k(G)$ denote the minimal eigenvalue of Δ_k. Note that $\mu_0(G)=\lambda_2(G)$. Our main result is the following

Theorem 1.1 *For $k\geq 1$*

$$k\mu_k(G)\geq (k+1)\mu_{k-1}(G)-n \quad . \tag{1}$$

As a direct consequence of Theorem 1.1 we obtain

Theorem 1.2 *If $\lambda_2(G)>\frac{kn}{k+1}$ then $\tilde{H}^k(X(G),\mathbb{R})=0$.*

Remarks:

1. Theorem 1.2 is related to a well-known result of Garland (Theorem 5.9

in [7]) and its extended version by Ballmann and Świątkowski (Theorem 2.5 in [5]). Roughly speaking, these results (in their simplest untwisted form) guarantee the vanishing of $\tilde{H}^k(X;\mathbb{R})$ provided that for *each* $(k-1)$-simplex τ in X, the spectral gap of the 1-skeleton of the link of τ is sufficiently large. Theorem 1.2 is, in a sense, a global counterpart of this statement for flag complexes.

2. Let $n = r\ell$ where $r \geq 1, \ell \geq 2$, and let G be the Turán graph $T_r(n)$, i.e. the complete r-partite graph on n vertices with all sides equal to ℓ. The flag complex $X(T_r(n))$ is homotopy equivalent to the wedge of $(\ell-1)^r$ $(r-1)$-dimensional spheres. It can be checked that $\mu_k(T_r(n)) = \ell(r-k-1)$ for all $0 \leq k \leq r-1$, hence (1) is satisfied with equality. Furthermore, $\lambda_2(G) = \ell(r-1) = \frac{r-1}{r}n$ while $\tilde{H}^{r-1}(X(G)) \neq 0$. Therefore the assumption in Theorem 1.2 cannot be replaced by $\lambda_2(G) \geq \frac{kn}{k+1}$.

We next study some graph theoretical consequences of Theorem 1.2. The *Independence Complex* $\mathrm{I}(G)$ of G is the simplicial complex on the vertex set V whose simplices are all independent sets $\sigma \subset V$. Thus $\mathrm{I}(G) = X(\overline{G})$ where $\overline{G}$ denotes the complement of G. Recent work on hypergraph matching, starting in [4] with later developments in [1, 10, 2, 3, 11], has utilized topological properties of $\mathrm{I}(G)$ to derive new Hall type theorems for hypergraphs. The main ingredient in these developments are lower bounds on the homological connectivity of $\mathrm{I}(G)$. For a simplicial complex Z let $\eta(Z) = \min\{i : \tilde{H}^i(Z,\mathbb{R}) \neq 0\} + 1$. It turns out that various domination parameters of G may be used to provide lower bounds on $\eta(\mathrm{I}(G))$. For a subset of vertices $S \subset V$ let $N(S)$ denote all vertices that are adjacent to at least one vertex of S and let $N'(S) = S \cup N(S)$. S is a *dominating set* if $N'(S) = V$. S is a *totally dominating set* if $N(S) = V$. Here are a few domination parameters:

- The *domination number* $\gamma(G)$ is the minimal size of a dominating set.
- The *total domination number* $\tilde{\gamma}(G)$ is the minimal size of a totally dominating set.
- The *independent domination number* $i\gamma(G)$ is the maximum, over all independent sets I in G, of the minimal size of a set S such that $N(S) \supset I$.

- The *strong fractional domination number*, $\gamma_s^*(G)$ is the minimum of $\sum_{v \in V} f(v)$, over all nonnegative functions $f : V \to \mathbb{R}$ such that $\sum_{uv \in E} f(u) + \deg(v) f(v) \geq 1$ for every vertex v.

Some known lower bounds on η are: $\eta(\mathrm{I}(G)) \geq \tilde{\gamma}(G)/2$ [10], $\eta(\mathrm{I}(G)) \geq i\gamma(G)$ [4], $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$ [11].

Here we introduce a new domination parameter, defined by vector representations. It is similar in spirit to the Θ function defined by Lovász [9]. It uses vectors to mimick domination, in a way similar to that in which the Θ function mimicks independence of sets of vertices. It is defined as follows. A *vector representation* of a graph $G = (V, E)$ is an assignment P of a vector $P(v) \in \mathbb{R}^\ell$ for some fixed ℓ to every vertex v of the graph, such that the inner product $P(u) \cdot P(v) \geq 1$ whenever u, v are adjacent in G and $P(u) \cdot P(v) \geq 0$ if they are not adjacent. We shall identify the representation with the matrix P whose v-th row is the vector $P(v)$.
Let $\mathbf{1}$ denote the all 1 vector in $\mathbb{R}^V$. A non-negative vector α on V is said to be *dominating for P* if $\sum_{v \in V} \alpha(v) P(v) \cdot P(u) \geq 1$ for every vertex u, namely $\alpha P P^T \geq \mathbf{1}$. (Note that taking α to be the characteristic function of some totally dominating set satisfies this condition regardless of the representation.) The *value* of P is

$$|P| = \min\{\alpha \cdot \mathbf{1} \;:\; \alpha \geq 0 \;,\; \alpha P P^T \geq \mathbf{1} \;\} \quad .$$

The supremum of $|P|$ over all vector representations P of G is denoted by $\Gamma(G)$. Our main application of Theorem 1.2 is the following

Theorem 1.3 $\eta(\mathrm{I}(G)) \geq \Gamma(G)$.

Remark: One natural vector representation of G is obtained by taking $P(v) \in \mathbb{R}^E$ to be the edge incidence vector of the vertex v. For this representation $|P| = \gamma_s^*(G)$ hence $\Gamma(G) \geq \gamma_s^*(G)$. The bound $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$ was previously obtained in [11]. Theorem 1.3 is however stronger and often gives much sharper estimates for $\eta(\mathrm{I}(G))$, see e.g. the case of cycles described in Section 4.

We next use Theorem 1.3 to derive a new Hall type result for hypergraphs. Let $\mathcal{F} \subset 2^V$ be a hypergraph on a finite ground set V. The *width* $w(\mathcal{F})$ of

$\mathcal{F}$ is the minimal t for which there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{F}$, $F_i \cap F \neq \emptyset$ for some $1 \leq i \leq t$.
The *fractional width* $w^*(\mathcal{F})$ of $\mathcal{F}$ is the minimum of $\sum_{E \in \mathcal{F}} f(E)$ over all non-negative functions $f : \mathcal{F} \to \mathbb{R}$ with the property that for every edge $E \in \mathcal{F}$ the sum $\sum_{F \in \mathcal{F}} f(F)|E \cap F|$ is at least 1. A *matching* in $\mathcal{F}$ is a subhypergraph $\mathcal{M} \subset \mathcal{F}$ such that $F \cap F' = \emptyset$ for all $F \neq F' \in \mathcal{M}$. Let $\{\mathcal{F}_i\}_{i=1}^m$ be a family of hypergraphs. A *system of disjoint representatives (SDR)* of $\{\mathcal{F}_i\}_{i=1}^m$ is a matching $F_1, \ldots, F_m$ such that $F_i \in \mathcal{F}_i$ for $1 \leq i \leq m$. Haxell [8] proved the following

Theorem 1.4 *[8] If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w(\cup_{i \in I} \mathcal{F}_i) \geq 2|I| - 1$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Here we use Theorem 1.3 to show

Theorem 1.5 *If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w^*(\cup_{i \in I} \mathcal{F}_i) > |I| - 1$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

The paper is organized as follows. In section 2 we recall some topological terminology and the simplicial Hodge theorem. Theorems 1.1 and 1.2 are proved in section 3. The proofs utilize the approach of Garland [7] and its exposition by Ballmann and Świątkowski [5]. In section 4 we relate the Γ parameter to homological connectivity and prove Theorem 1.3. In section 5 we recall a homological Hall type condition (Proposition 5.1) for the existence of colorful simplices in a colored complex. Combining this condition with Theorem 1.3 then completes the proof of Theorem 1.5.

2 Topological Preliminaries

Let X be a finite simplicial complex on the vertex set V. Let $X(k)$ denote the set of k-dimensional simplices in X, each taken with an arbitrary but fixed orientation. A simplicial k-cochain is a real valued skew-symmetric function on all ordered k-simplices of X. For $k \geq 0$ let $C^k(X)$ denote the space of k-cochains on X. The i-face of an ordered $(k+1)$-simplex $\sigma = [v_0, \ldots, v_{k+1}]$ is the ordered k-simplex $\sigma_i = [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]$. The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

It will be convenient to augment the cochain complex $\{C^i(X)\}_{i=0}^{\infty}$ with the (-1)-degree term $C^{-1}(X) = \mathbb{R}$ with the coboundary map $d_{-1} : C^{-1}(X) \to C^0(X)$ given by $d_{-1}(a)(v) = a$ for $a \in \mathbb{R}$, $v \in V$. Let $Z^k(X) = \ker(d_k)$ denote the space of k-cocycles and let $B^k(X) = \mathrm{Im}(d_{k-1})$ denote the space of k-coboundaries. For $k \geq 0$ let $\tilde{H}^k(X) = Z^k(X)/B^k(X)$ denote the k-th reduced cohomology group of X with real coefficients. For each $k \geq -1$ endow $C^k(X)$ with the standard inner product $(\phi, \psi) = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma)$ and the corresponding L^2 norm $||\phi|| = (\sum_{\sigma \in X(k)} \phi(\sigma)^2)^{1/2}$.
Let $d_k^* : C^{k+1}(X) \to C^k(X)$ denote the adjoint of d_k with respect to these standard inner products. The reduced k-Laplacian of X is the mapping

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X) \quad .$$

Note that if G denotes the 1-skeleton of X and J is the $V \times V$ all ones matrix, then the matrix $J + L_G$ represents Δ_0 with respect to the standard basis. In particular, the minimal eigenvalue of Δ_0 equals $\lambda_2(G)$.
The space of harmonic k-cochains $\tilde{\mathcal{H}}^k(X) = \ker \Delta_k$ consists of all $\phi \in C^k(X)$ such that both $d_k\phi$ and $d_{k-1}^*\phi$ are zero. The simplicial version of Hodge Theorem is the following well-known

Proposition 2.1 *$\tilde{\mathcal{H}}^k(X) \cong \tilde{H}^k(X)$ for $k \geq 0$.*

In particular, $\tilde{H}^k(X) = 0$ iff the minimal eigenvalue of Δ_k is positive.

3 Eigenvalues of Higher Laplacians

Let $X = X(G)$ be the flag complex of a graph $G = (V, E)$ on $|V| = n$ vertices. For an i-simplex $\eta \in X$ let $\deg(\eta)$ denote the number of $(i+1)$-simplices in X which contain η. The *link* of a simplex $\sigma \in X$ is the complex

$$\mathrm{lk}(\sigma) = \{\tau \in X \ : \ \sigma \cup \tau \in X \ , \ \sigma \cap \tau = \emptyset \ \} \ .$$

For two ordered simplices $\sigma \in X$, $\tau \in \mathrm{lk}(\sigma)$ let $\sigma\tau$ denote their ordered union.

Claim 3.1 *For $\phi \in C^k(X)$*

$$||d_k\phi||^2 = \sum_{\sigma \in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \mathrm{lk}(\eta)} \phi(v\eta)\phi(w\eta) \quad .$$

Proof: Recall that for $\tau \in X(k+1)$ we denoted by τ_i the ordered k-simplex obtained by removing the i-th vertex of τ. Thus

$$||d_k\phi||^2 = \sum_{\tau\in X(k+1)} d_k\phi(\tau)^2 = \sum_{\tau\in X(k+1)} \sum_{i=0}^{k+1}(-1)^i\phi(\tau_i)\sum_{j=0}^{k+1}(-1)^j\phi(\tau_j) =$$

$$\sum_{\tau\in X(k+1)} \sum_{i=0}^{k+1}\phi(\tau_i)^2 + \sum_{\tau\in X(k+1)} \sum_{i\neq j}(-1)^{i+j}\phi(\tau_i)\phi(\tau_j) =$$

$$\sum_{\sigma\in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2\sum_{\eta\in X(k-1)}\sum_{vw\in \mathrm{lk}(\eta)} \phi(v\eta)\phi(w\eta) \ \ .$$

$\square$

For $\phi \in C^k(X)$ and a vertex $u \in V$ define $\phi_u \in C^{k-1}(X)$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \in \mathrm{lk}(\tau) \\ 0 & \text{otherwise} \end{cases}$$

Claim 3.2 *For $\phi \in C^k(X)$*

$$\sum_{u\in V} ||d_{k-1}\phi_u||^2 =$$

$$\sum_{\sigma\in X(k)} (\sum_{\tau\in\sigma(k-1)} \deg(\tau))\phi(\sigma)^2 - 2k\sum_{\tau\in X(k-1)}\sum_{vw\in \mathrm{lk}(\tau)} \phi(v\tau)\phi(w\tau) \ .$$

Proof: Applying Claim 3.1 with $\phi_u \in C^{k-1}(X)$ we obtain

$$||d_{k-1}\phi_u||^2 = \sum_{\tau\in X(k-1)} \deg(\tau)\phi_u(\tau)^2 - 2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(\eta)} \phi_u(v\eta)\phi_u(w\eta) \ .$$

Hence

$$\sum_{u\in V} ||d_{k-1}\phi_u||^2 =$$

$$\sum_{u\in V}\sum_{\tau\in X(k-1)} \deg(\tau)\phi_u(\tau)^2 - 2\sum_{u\in V}\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(\eta)} \phi_u(v\eta)\phi_u(w\eta) =$$

$$\sum_{\sigma\in X(k)}(\sum_{\tau\in\sigma(k-1)}\deg(\tau))\phi(\sigma)^2-2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(\eta)}\sum_{u\in \mathrm{lk}(v\eta)\cap \mathrm{lk}(w\eta)}\phi(vu\eta)\phi(wu\eta)=$$

$$\sum_{\sigma\in X(k)}(\sum_{\tau\in\sigma(k-1)}\deg(\tau))\phi(\sigma)^2-2k\sum_{\tau\in X(k-1)}\sum_{vw\in \mathrm{lk}(\tau)}\phi(v\tau)\phi(w\tau)\ .$$

The last equality follows from the fact that since X is a flag complex, if $\eta\in X(k-2)$, $vw\in \mathrm{lk}(\eta)$ and $u\in \mathrm{lk}(v\eta)\cap \mathrm{lk}(w\eta)$, then $vw\in \mathrm{lk}(u\eta)$.

□

Claims 3.1 and 3.2 imply

$$k(||d_k\phi||^2-\sum_{\sigma\in X(k)}\deg(\sigma)\phi(\sigma)^2)=$$

$$\sum_{u\in V}||d_{k-1}\phi_u||^2-\sum_{\sigma\in X(k)}(\sum_{\tau\in\sigma(k-1)}\deg(\tau))\phi(\sigma)^2\quad . \tag{2}$$

Claim 3.3 *For $\phi\in C^k(X)$*

$$\sum_{u\in V}||d_{k-2}^*\phi_u||^2=k||d_{k-1}^*\phi||^2\quad . \tag{3}$$

Proof: For $\tau\in X(k-1)$

$$d_{k-1}^*\phi(\tau)=\sum_{v\in \mathrm{lk}(\tau)}\phi(v\tau)\ .$$

Therefore

$$||d_{k-1}^*\phi||^2=\sum_{\tau\in X(k-1)}d_{k-1}^*\phi(\tau)^2=$$

$$\sum_{\tau\in X(k-1)}(\sum_{v\in \mathrm{lk}(\tau)}\phi(v\tau))(\sum_{w\in \mathrm{lk}(\tau)}\phi(w\tau))=\sum_{\tau\in X(k-1)}\sum_{(v,w)\in \mathrm{lk}(\tau)^2}\phi(v\tau)\phi(w\tau)\ . \tag{4}$$

Substituting ϕ_u in (4) we obtain

$$\sum_{u\in V}||d_{k-2}^*\phi_u||^2=\sum_{u\in V}\sum_{\eta\in X(k-2)}\sum_{(v,w)\in \mathrm{lk}(\eta)^2}\phi_u(v\eta)\phi_u(w\eta)\ =$$

$$\sum_{\eta\in X(k-2)}\sum_{u\in \mathrm{lk}(\eta)}\sum_{(v,w)\in \mathrm{lk}(u\eta)^2}\phi(vu\eta)\phi(wu\eta) =$$

$$k\sum_{\tau\in X(k-1)}\sum_{(v,w)\in \mathrm{lk}(\tau)^2}\phi(v\tau)\phi(w\tau) = k||d_{k-1}^*\phi||^2 .$$

$\square$

Let $\phi \in C^k(X)$. Summing (2) and (3) we obtain the following key identity:

$$k(\Delta_k\phi,\phi) =$$

$$\sum_{u\in V}(\Delta_{k-1}\phi_u,\phi_u) - \sum_{\sigma\in X(k)}(\sum_{\tau\in\sigma(k-1)}\deg(\tau) - k\deg(\sigma))\phi(\sigma)^2 \quad . \tag{5}$$

To estimate the righthand side of (5) we need the following

Claim 3.4 *For $\sigma \in X(k)$*

$$\sum_{\tau\in\sigma(k-1)}\deg(\tau) - k\deg(\sigma) \le n . \tag{6}$$

Proof: Recall that $N(v)$ is the set of neighbors of v in G. Let $\sigma = [v_0,\dots,v_k]$ then for any $I \subset \{0,\dots,k\}$

$$\deg([v_i : i \in I]) = |\bigcap_{i\in I}N(v_i)| \quad .$$

Therefore

$$\sum_{\tau\in\sigma(k-1)}\deg(\tau) - k\deg(\sigma) = \sum_{i=0}^{k}|\bigcap_{j\neq i}N(v_j)| - k|\bigcap_{j=0}^{k}N(v_j)| \quad . \tag{7}$$

The Claim now follows since each $v \in V$ is counted at most once on the righthand side of (7).

$\square$

Proof of Theorem 1.1: Let $0 \neq \phi \in C^k(X)$ be an eigenvector of Δ_k with eigenvalue $\mu_k(G)$. By double counting

$$\sum_{u \in V} ||\phi_u||^2 = (k+1)||\phi||^2 \quad . \tag{8}$$

Combining (5),(6) and (8) we obtain

$$k\mu_k(G)||\phi||^2 = k(\Delta_k \phi, \phi) \geq \sum_{u \in V} (\Delta_{k-1}\phi_u, \phi_u) - n \sum_{\sigma \in X(k)} \phi(\sigma)^2 \quad \geq$$

$$\mu_{k-1}(G) \sum_{u \in V} ||\phi_u||^2 - n||\phi||^2 = ((k+1)\mu_{k-1}(G) - n)||\phi||^2 \quad .$$

□

Proof of Theorem 1.2: Inequality (1) implies by induction on k that $\mu_k(G) \geq (k+1)\mu_0(G) - kn$. Therefore, if $\mu_0(G) = \lambda_2(G) > \frac{kn}{k+1}$ then $\mu_k(G) > 0$ and $\tilde{H}^k(X(G), \mathbb{R}) = 0$ follows from the simplicial Hodge Theorem.

□

4 Vector Domination and Homology

Let $G = (V, E)$ be a graph with $|V| = n$. We first reformulate Theorem 1.2 in terms of the independence complex $\mathrm{I}(G)$.

Theorem 4.1 $\eta(\mathrm{I}(G)) \geq \frac{n}{\lambda_n(G)}$.

Proof: Let $\ell = \lceil \frac{n}{\lambda_n(G)} \rceil$. Since $\lambda_n(G) = n - \lambda_2(\overline{G})$ it follows that $\lambda_2(\overline{G}) > \frac{\ell-2}{\ell-1} n$. Therefore by Theorem 1.2, $\tilde{H}^i(\mathrm{I}(G)) = \tilde{H}^i(X(\overline{G})) = 0$ for $i \leq \ell - 2$. Hence $\eta(\mathrm{I}(G)) \geq \ell$.

□

The proof of Theorem 1.3 depends on Theorem 4.1 and the following

Claim 4.2 *Let P be a vector representation of $G = (V, E)$. Then*

$$\lambda_n(G) \leq \max_{u \in V} \ P(u) \cdot \sum_{v \in V} P(v) \quad .$$

Proof: Let $x = (x(v) : v \in V)$ be a vector in $\mathbb{R}^V$. Then

$$x^T L_G x = \sum_{uv \in E} (x(u) - x(v))^2 \leq$$

$$\frac{1}{2} \sum_{(u,v) \in V \times V} (x(u) - x(v))^2 P(u) \cdot P(v) =$$

$$\sum_{u \in V} x(u)^2 P(u) \cdot \sum_{v \in V} P(v) \ - \ ||\sum_{v \in V} x(v) P(v)||^2 \leq$$

$$||x||^2 \ \max_{u \in V} \ P(u) \cdot \sum_{v \in V} P(v) \ .$$

The Claim follows since $\lambda_n(G) = \max \ \{\frac{x^T L_G x}{||x||^2} \ : \ 0 \neq x \in \mathbb{R}^V\}$.

$\square$

Let $\mathbb{Z}_+$ denote the positive integers and let $\mathbb{Q}_+$ denote the positive rationals. For a vector $\mathbf{a} = (a(v) : v \in V) \in \mathbb{Z}_+^V$ let $G_{\mathbf{a}}$ denote the graph obtained by replacing each $v \in V$ by an independent set of size $a(v)$. Formally $V(G_{\mathbf{a}}) = \{(v, i) : v \in V \ , \ 1 \leq i \leq a(v)\}$ and $\{(u, i), (v, j)\} \in E(G_{\mathbf{a}})$ if $\{u, v\} \in E$. The projection $(v, i) \to v$ induces a homotopy equivalence between $\mathrm{I}(G_{\mathbf{a}})$ and $\mathrm{I}(G)$. In particular $\eta(\mathrm{I}(G_{\mathbf{a}})) = \eta(\mathrm{I}(G))$.

Proof of Theorem 1.3: Let P be a representation of G. By linear programming duality

$$|P| = \min\{\alpha \cdot \mathbf{1} \ : \ \alpha \geq 0 \ , \ \alpha P P^T \geq \mathbf{1} \ \} =$$

$$\max\{\alpha \cdot \mathbf{1} \ : \ \alpha \geq 0 \ , \ \alpha P P^T \leq \mathbf{1} \ \} =$$

$$\mathrm{Sup}\{\alpha \cdot \mathbf{1} \ : \ \alpha \in \mathbb{Q}_+^V \ \ , \ \ \alpha P P^T \leq \mathbf{1} \ \} \ .$$

Let $\alpha \in \mathbb{Q}_+^V$ such that $\alpha PP^T \leq \mathbf{1}$. Write $\alpha = \frac{\mathbf{a}}{k}$ where $k \in \mathbb{Z}_+$ and $\mathbf{a} = (a(v) : v \in V) \in \mathbb{Z}_+^V$. Let $N = |V(G_\mathbf{a})| = \sum_{u \in V} a(u)$. Consider the representation Q of $G_\mathbf{a}$ given by $Q((u,i)) = P(u)$ for $(u,i) \in V(G_\mathbf{a})$. By Claim 4.2

$$\lambda_N(G_\mathbf{a}) \leq \max_{(u,i)\in V(G_\mathbf{a})} Q((u,i)) \cdot \sum_{(v,j)\in V(G_\mathbf{a})} Q((v,j)) =$$

$$\max_{u \in V} P(u) \cdot \sum_{v \in V} a(v)P(v) \leq k \quad .$$

Hence by Theorem 4.1

$$\alpha \cdot \mathbf{1} = \frac{1}{k} \sum_{v \in V} a(v) = \frac{N}{k} \leq$$

$$\frac{N}{\lambda_N(G_\mathbf{a})} \leq \eta(\mathrm{I}(G_\mathbf{a})) = \eta(\mathrm{I}(G)) \quad .$$

□

Remarks:

1. Let C_n denote the n-cycle on the vertex set $V = \{0, \ldots, n-1\}$. For $n = 3k$ define a representation P of C_{3k} by

$$P(\ell) = \begin{cases} e_{2j} & \ell = 3j \\ e_{2j} + e_{2j+1} & \ell = 3j+1 \\ e_{2j+1} + e_{2j+2} & \ell = 3j+2 \end{cases}$$

where $e_0, \ldots, e_{2k-1}$ are orthogonal unit vectors and the indices are cyclic modulo $2k$. Let $\alpha \in \mathbb{R}^V$ be given by $\alpha(\ell) = 1$ if 3 divides ℓ and zero otherwise. Since $\alpha PP^T = \mathbf{1}$, it follows by linear programming duality that $\Gamma(C_{3k}) \geq \alpha \cdot \mathbf{1} = k$. On the other hand (see Claim 3.3 in [11]) $\eta(I(C_n)) = \lfloor \frac{n+1}{3} \rfloor$. Therefore $\eta(\mathrm{I}(C_{3k})) = \Gamma(C_{3k}) = k$. For $n = 3k+1$ it can similarly be shown that $\eta(\mathrm{I}(C_{3k+1})) = \Gamma(C_{3k+1}) = k$. The case $n = 3k-1$ is more involved and we only have the bounds $k - \frac{1}{2} \leq \Gamma(C_{3k-1}) \leq \eta(\mathrm{I}(C_{3k-1})) = k$. Note that for cycles the bound $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$ is weaker since $\gamma_s^*(C_n) = \frac{n}{4}$.

2. It can be shown that for any graph $\Gamma(G) \geq \mathrm{Sup}\{\gamma_s^*(G_\mathbf{a}) \; : \; \mathbf{a} \in \mathbb{Z}_+^V\}$. We do not know of examples with strict inequality.

5 A Hall Type Theorem for Fractional Width

Let Z be a simplicial complex on the vertex set W and let $\bigcup_{i=1}^m W_i$ be a partition of W. A simplex $\tau \in Z$ is *colorful* if $|\tau \cap W_i| = 1$ for all $1 \leq i \leq m$. For $W' \subset W$ let $Z[W']$ denote the induced subcomplex on W'. The following Hall's type sufficient condition for the existence of colorful simplices appears in [4] and in [10].

Proposition 5.1 *If for all $\emptyset \neq I \subset [m]$*

$$\eta(Z[\bigcup_{i \in I} W_i]) \geq |I|$$

then Z contains a colorful simplex.

Let G be a graph on the vertex set W with a partition $W = \bigcup_{i=1}^m W_i$. A set $S \subset W$ is *colorful* if $S \cap W_i \neq \emptyset$ for all $1 \leq i \leq m$. The induced subgraph on $W' \subset W$ is denoted by $G[W']$. Combining Theorem 1.3 and Proposition 5.1 we obtain the following

Theorem 5.2 *If $\Gamma(G[\bigcup_{i \in I} W_i]) > |I| - 1$ for all $\emptyset \neq I \subset [m]$ then G contains a colorful independent set.*

Let $\mathcal{F} \subset 2^V$ be a hypergraph, possibly with multiple edges. The *line graph* $G_{\mathcal{F}} = (W, E)$ associated with $\mathcal{F}$ has vertex set $W = \mathcal{F}$ and edge set E consisting of all $\{F, F'\} \subset \mathcal{F}$ such that $F \cap F' \neq \emptyset$. A matching in $\mathcal{F}$ corresponds to an independent set in $G_{\mathcal{F}}$. For each $F \in \mathcal{F}$ let $P(F) \in \mathbb{R}^V$ denote the incidence vector of F. P is clearly a vector representation of $G_{\mathcal{F}}$ and satisfies $|P| = w^*(\mathcal{F})$. Thus $\Gamma(G_{\mathcal{F}}) \geq w^*(\mathcal{F})$.

Proof of Theorem 1.5: Let $\mathcal{F}$ denote the disjoint union of the $\mathcal{F}_i$'s, and consider the graph $G_{\mathcal{F}} = (W, E)$ with the partition $W = \cup_{i=1}^m W_i$ where $W_i = \mathcal{F}_i$. Then for any $\emptyset \neq I \subset [m]$

$$\Gamma(G_{\mathcal{F}}[\cup_{i \in I} W_i]) = \Gamma(G_{\cup_{i \in I} \mathcal{F}_i}) \geq$$

$$w^*(\cup_{i \in I} \mathcal{F}_i) > |I| - 1 \quad .$$

Theorem 5.2 implies that $G_{\mathcal{F}}$ contains a colorful independent set, hence $\{\mathcal{F}_i\}_{i=1}^m$ contains an SDR.

□

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